ON THE MOTION OF A PISTON IN AN IDEAL GAS

(O DVIZHENII PORSHNIA V IDEAL'NOM GAZE)

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The solution of the problem of the expansion of a piston at constant speed in an ideal gas was given by Sedov [1,2] and Taylor [3]. If the speed of the piston is constant, the motion of the gas is self-similar, including the counter-pressure ($p_1 \neq 0$); if the piston radius varies with time according to a power law, the motion will be self-similar only if the counter-pressure is not included. Self-similar motions of a gas, produced by a piston moving according to a power law, were considered in [4-8]. The solution of a non-self-similar ($p_1 \neq 0$) linearized problem of the expansion of a piston with speed

$$v_n = ct^m \left[1 + \frac{(m-1)}{2(2m-1)} \left(\frac{\gamma p_1 \lambda_*}{\rho_1 c} \right)^2 A t^{-2m} \right]$$

was given in [9].

In the present paper, we consider the non-self-similar problem ($p_1 \neq 0$) of the motion of a gas due to a piston moving with the speed $v_n = ct^m$. These motions may be considered as being created from a point explosion with a diverging shock wave, or from a peripheral explosion, with converging shock wave, taking the gas to be pushed by the products of the explosion. It is assumed that the motion of the explosion products is similar to the motion of a piston following a power law.

1. We shall consider the nonstationary motion of a gas produced by a plane cylindrical or spherical piston, having an arbitrary motion. At the initial instant the gas is at rest; its density ρ_1 and pressure p_1 are uniform. For the independent variables and the unknown functions we shall take the dimensionless quantities

$$\lambda = \frac{r}{r_2}, \qquad q = \frac{a_1^2}{c^2}, \qquad f(\lambda, q) = \frac{v}{v_2}, \qquad R(\lambda, q) = \frac{\rho}{\rho_2}, \qquad P(\lambda, q) = \frac{p}{p_2}$$

where a_1 is the speed of sound in the undisturbed gas, r_2 is the radius of the shock wave, v_2, ρ_2, p_2 are, respectively, the speed, density and pressure behind the shock front. In these variables the equations of motion of the disturbed gas have the form

$$\left[\frac{2(1-q)f}{\gamma+1}-\lambda\right](1-q)\frac{\partial f}{\partial\lambda}+\frac{[2\gamma-(\gamma-1)q][\gamma-1+2q]}{2\gamma(\gamma+1)}\frac{1}{R}\frac{\partial P}{\partial\lambda}+ \\ +\left[(1-q)\frac{\partial f}{\partial\lambda}-\frac{(1+q)}{2q}f\right]\frac{dq}{ds}=0 \qquad (1.1)$$

$$\frac{2(1-q)}{\gamma+1}\frac{\partial f}{\partial\lambda}+\left[\frac{2(1-q)}{\gamma+1}f-\lambda\right]\frac{1}{R}\frac{\partial R}{\partial\lambda}+\left[\frac{1}{R}\frac{\partial R}{\partial q}-\frac{2}{\gamma-1+2q}\right]\frac{dq}{ds}+ \\ +\frac{2(\nu-1)(1-q)}{\gamma+1}\frac{f}{\lambda}=0 \qquad (1.2)$$

$$\left[\frac{2(1-q)}{\gamma+1}f-\lambda\right]\frac{1}{P}\frac{\partial P}{\partial\lambda}-\gamma\left[\frac{2(1-q)}{\gamma+1}f-\lambda\right]\frac{1}{R}\frac{\partial R}{\partial\lambda}+ \\ +\left[\frac{1}{P}\frac{\partial P}{\partial q}-\frac{\gamma}{R}\frac{\partial R}{\partial q}-\frac{2\gamma(\gamma-1)(1-q)^2}{(\gamma-1+2q)[2\gamma-(\gamma-1)q]q}\right]\frac{dq}{ds}=0 \qquad (1.3)$$

Here γ is the adiabatic index, and

$$s=\ln kr_2 \qquad (k=\text{const},\ [k]=\mathrm{L}^{-1}) \qquad (1.4)$$

The values ν = 1, 2, 3 correspond to plane, cylindrical and spherical waves.

Let the dimensionless piston coordinate and the dimensionless piston velocity depend on q as follows:

$$\lambda_n=\xi(q), \qquad f_n[\lambda_n, q]=\eta(q) \qquad (1.5)$$

The condition that the velocity of particles next to the piston is equal to the piston velocity is written as follows:

$$\frac{d\xi}{dq}+\xi\frac{ds}{dq}=\frac{2}{\gamma+1}(1-q)\eta\frac{ds}{dq} \qquad (1.6)$$

If one of the three functions $\xi(q)$, $\eta(q)$ or $s(q)$ is known, then to find the other two it is necessary to make use of Equations (1.5) and (1.6). If the law for the piston expansion, $r_n=\phi(t)$, is given as a function of time, Equation (1.6) takes the form

$$r_n=\frac{dr_n}{dt}=\varphi'(t) \qquad (1.7)$$

Using Equations (1.5) and (1.7), and the expression for the velocity of the shock wave and the velocity of the fluid particles behind it, we obtain

$$\xi(q)=\left(r_2(t_0)+a_1\int_{t_0}^{t}q^{-\frac{1}{2}}dt\right)^{-1}\varphi(t), \quad \eta(q)=\left(\frac{2}{\gamma+1}a_1q^{-\frac{1}{2}}(1-q)\right)^{-1}\varphi'(t) \qquad (1.8)$$

If one of the four functions $q(t)$, $\eta(q)$, $\xi(q)$, $\phi(t)$ is given, the others may be found from Equations (1.5) and (1.8). It should be noted that the formulation of the problem imposes certain restrictions on these functions.

The solution of the problem of a piston moving in an ideal gas according to the law (1.5) reduces to the integration of the nonlinear system of partial differential equations (1.1)-(1.3) in some region of the λ, q plane (with $0 \leqslant q \leqslant 1$), with the following conditions: at the shock wave

$$f(1, q) = R(1, q) = P(1, q) = 1 \qquad \text{for } \lambda = 1 \tag{1.9}$$

the condition (1.6) on the surface of the piston, with $\lambda = \xi(q)$, and certain initial conditions at $q = 0$.

2. Let the piston move according to a power law

$$v_n = ct^m \tag{2.1}$$

We shall consider motions which are nearly self-similar, and represent the functions s and r_2 in the form

$$s = \ln (A_0 q)^{\frac{1}{\nu_1}} + \frac{A}{\nu_1} q + O(q^2), \quad r_2 = \frac{1}{k}(A_0 q)^{\frac{1}{\nu_1}}\left[1 + \frac{A}{\nu_1} q\right] + O(q^2) \tag{2.2}$$

where

$$\nu_1 = -\frac{2m}{m+1}, \qquad A_0 = \frac{1}{\gamma}\left(\frac{(m+1)^m}{\lambda_*}\right)^{\frac{2}{m+1}}, \qquad k = \left[c\left(\frac{\rho_1}{p_1}\right)^{\frac{m+1}{2}}\right]^{\frac{1}{m}} \tag{2.3}$$

λ_* is the dimensionless piston coordinate, determined in the solution of the corresponding self-similar problem. A is a constant, unknown so far, which is determined from the solution.

Using Equations (1.4) and (2.2), we find the relation

$$t = \left[\frac{\lambda_*}{c}\left(\frac{\gamma p_1}{\rho_1}\right)^{\frac{1}{2}}\right]^{\frac{1}{m}} q^{-\frac{1}{2m}}\left[1 + \frac{(1-m)}{2m(2m-1)} Aq\right] \tag{2.4}$$

In view of (2.1)-(2.4), the expressions for the functions (1.5) take the following form:

$$\begin{aligned} \lambda_n &= \xi(q) = \lambda_* + aq \qquad \left(a = \frac{(m+1)\lambda_* A}{2(2m-1)}\right) \\ f_n &= \eta(q) = \frac{\gamma+1}{2}\lambda_*\left\{1 + \left[1 - \frac{(m-1)}{2(2m-1)} A\right] q\right\} \end{aligned} \tag{2.5}$$

The initial conditions for $q = 0$ may be written in the form

$$f(\lambda, 0) = f_0(\lambda), \qquad R(\lambda, 0) = R_0(\lambda), \qquad P(\lambda, 0) = P_0(\lambda) \tag{2.6}$$

where $f_0(\lambda)$, $R_0(\lambda)$, $P_0(\lambda)$ are functions corresponding to the self-similar motion [4-8]. In place of the variables λ and q we introduce x and q, with

$$x_* = \frac{\lambda - \lambda_* - aq}{1 - \lambda_* - aq} \tag{2.7}$$

We shall look for the functions $f(x, q)$, $R(x, q)$ and $P(x, q)$ in the form

$$\begin{gathered} f(x, q) = f_0(x) + qf_1(x) + \ldots \\ R(x, q) = R_0(x) + qR_1(x) + \ldots, \quad P(x, q) = P_0(x) + qP_1(x) + \ldots \end{gathered} \tag{2.8}$$

Neglecting terms of order q^2 and higher, we obtain for $f_1(x)$, $R_1(x)$, $P_1(x)$ and the constant A a linear system of differential equations (the primes denote differentiation with respect to x).

$$\begin{gathered} a_1 R_0 f_1' + \frac{\gamma - 1}{\gamma + 1} P_1' + a_2 f_1 + a_3 R_1 + b_{11} + Ab_{12} = 0 \\ \frac{2}{\gamma + 1} R_0 f_1' + a_1 R_1' + a_4 f_1 + a_5 R_1 + b_{21} + Ab_{22} = 0 \\ a_1 (R_0 P_1' - \gamma P_0 R_1') + a_6 f_1 - \gamma a_1 R_0' P_1 + a_7 R_1 + b_{31} + Ab_{32} = 0 \end{gathered} \tag{2.9}$$

Here

$$a_1 = \frac{2}{\gamma + 1} f_0 - \lambda_* - (1 - \lambda_*) x, \qquad a_2 = \left[\frac{2}{\gamma + 1} f_0' + \frac{\nu_1 (1 - \lambda_*)}{2}\right] R_0$$

$$a_3 = a_1 f_0' - \frac{\nu_1 (1 - \lambda_*)}{2} f_0, \qquad a_4 = \frac{2}{\gamma + 1}\left[\frac{(\nu - 1)(1 - \lambda_*) R_0}{\lambda_* + x(1 - \lambda_*)} + R_0'\right]$$

$$a_5 = \nu_1 (1 - \lambda_*) + \frac{2}{\gamma + 1}\left[f_0' + \frac{(\nu - 1)(1 - \lambda_*)}{\lambda_* + x(1 - \lambda_*)} f_0\right]$$

$$a_6 = \frac{2}{(\gamma + 1)} (R_0 P_0' - \gamma P_0 R_0'), \qquad a_7 = -\nu_1 (1 - \lambda_*)(\gamma + 1) P_0 + a_1 P_0'$$

$$\begin{gathered} b_{11} = \frac{\gamma^2 + 4\gamma - 1}{2\gamma(\gamma + 1)} P_0' - \left[\frac{2}{\gamma + 1} f_0' + \nu_1 (1 - \lambda_*)\right] f_0 R_0 \\ b_{21} = -\frac{2\nu_1 (1 - \lambda_*) R_0}{\gamma - 1} - [\lambda_* + x(1 - \lambda_*)] R_0' \\ b_{31} = -\frac{2}{\gamma + 1} (R_0 P_0' - \gamma P_0 R_0') f_0 + \nu_1 (1 - \lambda_*) \frac{[4\gamma^2 - (\gamma - 1)^2]}{2\gamma(\gamma - 1)} P_0 R_0 \\ b_{12} = \frac{\nu_1 (1 - \lambda_*)}{2} f_0 R_0 + \frac{(m + 1)\lambda_*}{2(2m - 1)} \left\{\frac{\nu_1}{2} f_0 + (\nu_1 + 1)(x - 1) f_0'\right\} R_0 \\ b_{22} = \frac{(m + 1)\lambda_*}{2(2m - 1)} \left\{(\nu_1 + 1)(x - 1) R_0' - \frac{2(\nu - 1) R_0 f_0}{(\gamma + 1)[\lambda_* + x(1 - \lambda_*)]^2}\right\} \\ b_{32} = \nu_1 (1 - \lambda_*) P_0 R_0 + \frac{(m + 1)\lambda_*}{2(2m - 1)} \{\nu_1 P_0 R_0 + (\nu_1 + 1)(x - 1)(R_0 P_0' - \gamma P_0 R_0')\} \end{gathered} \tag{2.10}$$

The coefficients of this system are known functions of x, expressed in terms of $f_0(x)$, $R_0(x)$, $P_0(x)$ and x; the functions $f_0(x)$, $R_0(x)$ and $P_0(x)$ are known from the solution of the self-similar problem. The conditions at the shock wave have the form

$$f_1(1) = R_1(1) = P_1(1) = 0 \tag{2.11}$$

(since for the self-similar functions we have $f_0(1) = R_0(1) = P_0(1) = 1$).

Relations (2.5)-(2.8), together with the conditions on the piston for the solution of the self-similar problem $f_0 = 1/2(\gamma + 1)\lambda_*$, give the boundary condition at points on the piston surface

$$f_1(0) = \frac{(\gamma+1)\lambda_*}{2}\left[1 - \frac{(m-1)A}{2(2m-1)}\right] \tag{2.12}$$

Thus, the problem reduces to the integration of the system of differential equations (2.9) on the interval $0 < x < 1$, with the boundary conditions (2.11) and (2.12). From the form of the equations and the boundary conditions, it is clear that the solution of this problem may be sought in the form [1]

$$f_1 = f_{11} + Af_{12}, \quad R_1 = R_{11} + AR_{12}, \quad P_1 = P_{11} + AP_{12} \tag{2.13}$$

Putting (2.13) in (2.9), we obtain two systems of differential equations, which must be satisfied by the functions f_{1i}, R_{1i} and P_{1i} $(i = 1, 2)$:

$$\begin{gathered} a_1 R_0 f_{1i}' + \frac{\gamma-1}{\gamma+1} P_{1i}' + a_2 f_{1i} + a_3 R_{1i} + b_{1i} = 0 \\ \frac{2R_0}{\gamma+1} f_{1i}' + a_1 R'_{1i} + a_4 f_{1i} + a_5 R_{1i} + b_{2i} = 0 \\ a_1 (R_0 P_{1i}' - \gamma P_0 R_{1i}') + a_6 f_{1i} - \gamma a_1 R_0' P_{1i} + a_7 R_{1i} + b_{3i} = 0 \end{gathered} \tag{2.14}$$

The coefficients of Equations (2.14) are determined by Equations (2.10).

In view of (2.13), (2.11), (2.12), the boundary conditions for f_{1i}, R_{1i}, P_{1i} take the form

$$f_{1i}(1) = R_{1i}(1) = P_{1i}(1) = 0 \tag{2.15}$$

$$f_{11}(0) + Af_{12}(0) = \frac{(\gamma+1)\lambda_*}{2}\left[1 - \frac{(m-1)}{2(2m-1)} A\right] \tag{2.16}$$

Condition (2.16) is used to find the constant A after the functions f_{1i}, R_{1i}, P_{1i} are found.

3. We note that Equations (2.9) have an adiabatic integral, analogous to that found in [9]

$$\frac{2}{(\gamma+1)}\frac{\nu_1}{(\nu_1+\nu)}\frac{f_1}{\left[\lambda_*+x(1-\lambda_*)-\frac{2}{\gamma+1}f_0\right]}-\left(\frac{\nu}{\nu_1+\nu}-\gamma\right)\frac{R_1}{R_0}-\frac{P_1}{P_0}=$$

$$=\frac{R_0^{\gamma}}{P_0}\int_1^x\frac{P_0}{R_0^{\gamma}}\frac{F(x)\,dx}{\left[\lambda_*+x(1-\lambda_*)-\frac{2}{\gamma+1}f_0\right]} \qquad (3.1)$$

Here

$$F(x)=F_1(x)+AF_2(x) \qquad (3.2)$$

$$F_1(x)=[\lambda_*+x(1-\lambda_*)]\frac{P_0'}{P_0}-\left(\frac{\nu_1}{\nu_1+\nu}-\gamma\right)\frac{b_{21}}{R_0}+\frac{\nu_1(1-\lambda_*)(3\gamma-1)}{2\gamma}$$

$$F_2(x)=-\frac{\nu_1 b_{22}}{(\nu+\nu_1)R_0}-\frac{(m+1)\nu_1\lambda_*}{2(2m-1)}\left[1+\frac{(\nu_1+1)(x-1)(1-\lambda_*)}{a_1}\right]-\nu_1(1-\lambda_*)$$

4. Near the piston, where $x = 0$, the solution of the self-similar problem f_0, R_0 and P_0 has singularities; therefore, for the integration of Equations (2.9), it is necessary to make use of asymptotic formulas.

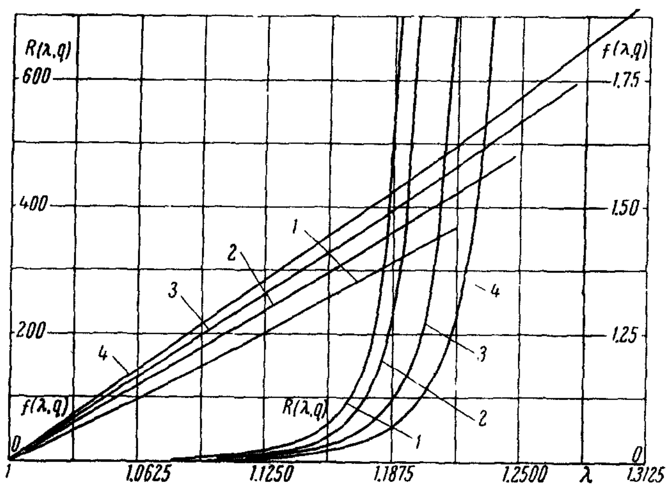

FIG. 1.

We shall take $m = -\nu/(2+\nu)$; in this case, in the zV plane, where $z = a^2t^2/r^2$, $V = vt/r$, the field of the integral curves of Equation (1.4) of [8], coincides with the field of the integral curves for a strong explosion [1].

Depending on the parameters γ and ν appearing in Equation (1.4), three cases are possible [8]:

1) for $\gamma < 2$ the solution of the self-similar problem of the piston coincides with the corresponding solution for the problem of a peripheral

explosion [5];

2) for $\gamma > 2$, $\nu = 1$ or $\nu = 2$, and also for $2 < \gamma < 7$, $\nu = 3$, a solution of the self-similar piston problem does not exist;

3) for $\gamma > 7$, $\nu = 3$ the solution of the self-similar piston problem coincides with the corresponding solution of the problem of a strong explosion.

It follows that the solutions of the self-similar piston problem for the case $m = -\nu/(2+\nu)$ $(\nu_1 = \nu)$ is described by Equations (11.15)-(11.16), obtained by Sedov [1], where $4/(2+\nu)(\gamma+1) \leqslant V \leqslant 2/(2+\nu)$. If we introduce a new independent variable

$$u = \frac{2}{2+\nu} - V \tag{4.1}$$

and make use of the solution of Sedov [1], we find the asymptotic behavior of the functions $f_0(u)$, $R_0(u)$, $P_0(u)$ in the neighborhood of $u = 0$:

$$f_0 = \frac{\gamma+1}{2}\lambda_*, \qquad R_0 = C_1 u^{\frac{2}{\gamma-2}}, \qquad P_0 = C_2 u^{\frac{\gamma}{\gamma-2}} \tag{4.2}$$

$$C_1 = (\gamma+1)^{\alpha_3}\left(\frac{\gamma+1}{\gamma-1}\,\frac{2+\nu}{2}\right)^{\alpha_5}\left[\frac{\nu(\gamma+1)(2-\gamma)}{2\gamma-\nu\gamma+3\nu-2}\right]^{\alpha_4}, \qquad C_2 = \frac{(\gamma+1)^2(2+\nu)\lambda_*^2}{8(\gamma-1)}C_1$$

where α_3, α_4, α_5 are known functions of ν and γ [1].

Putting the expressions for the functions f_0, R_0, P_0 and their derivatives in the first two equations (2.14), and using the adiabatic integral (3.1), we obtain a system of two linear differential equations for determining $f_{1i}(u)$ and $R_{1i}(u)$ in the neighborhood of the piston $(\lambda = \lambda_*)$. Integrating this system, taking into account the integral (3.1), we obtain asymptotic representations for the functions $f_{1i}(u)$, $R_{1i}(u)$ and $P_{1i}(u)$:

$$f_{1i} = C_{4i}\,u^{\frac{2}{2-\gamma}} - \frac{2+\nu}{4}\,\frac{(\gamma^2-1)\lambda_* C_{3i}}{C_1}, \quad R_{1i} = C_{3i}\,u^{\frac{4-\gamma}{\gamma-2}} + \frac{8}{2+\nu}\,\frac{C_1{}^{\gamma}Y_{1i}}{(\gamma+1)^2\lambda_*^2}\,u^{\frac{\gamma+2}{\gamma-2}}$$

$$P_{1i} = \frac{2+\nu}{16}\,\frac{\gamma(\gamma+1)^2\lambda_*^2 C_{3i}}{(\gamma-1)}\,u^{\frac{2}{\gamma-2}} + \frac{C_1{}^{\gamma}Y_{1i}}{2(\gamma-1)}\,u^{\frac{2\gamma}{\gamma-2}} + \frac{(\gamma+1)\lambda_* C_1 C_{4i}}{4(\gamma-1)} \tag{4.3}$$

Here

$$Y_{1i} = \int_0^1 \frac{P_0}{R_0{}^{\gamma}}\,\frac{F_i(x)\,dx}{\left[\lambda_* + x(1-\lambda_*) - \frac{2}{\gamma+1}f_0(x)\right]}, \quad (\gamma < 2) \tag{4.4}$$

The functions $F_i(x)$ are determined from Equations (3.2), C_1 from Equation (4.2), C_{3i}, C_{4i} are constants of integration.

Making use of Equation (2.16) and the first of Equations (4.3), we find the following expression for the constant A:

$$A = \frac{C_1 + 2(\gamma - 1) C_{31}}{\frac{m-1}{2(2m-1)} C_1 - 2(\gamma - 1) C_{32}} \tag{4.5}$$

Figures 1 and 2 give, for various q, the variation of the characteristics of the motion, namely the velocity, density and pressure in air ($\gamma = 1.4$), between a cylindrical piston and an imploding shock wave. Curves marked 1 describe self-similar motion ($q = 0$), curves 2, 3, 4 correspond to the values $q = 0.025,\ 0.050,\ 0.075$.

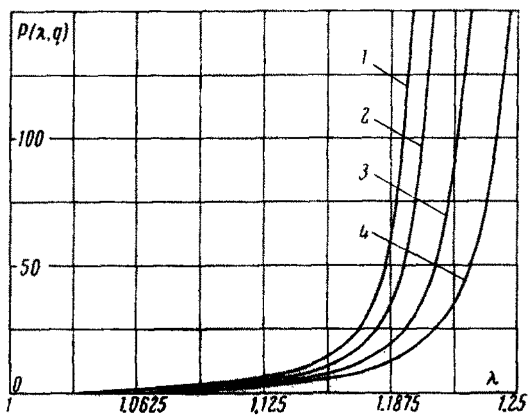

FIG. 2.

According to the hypothesis of plane cross-sections [11], this will be the variation of the characteristics of the motion in the case of flow at high but finite velocity over an axisymmetric body in a duct. The form of the body is nearly parabolic (the self-similar problem for a body of parabolic form $r = cx^{0.5}$ is considered in [5]).

We note that the problem of the outward propagating shock wave, where the piston expands according to a power law, is considered in [10].

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